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# Dense embeddings in pathwise connected spaces

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**Abstract**

This paper is devoted to the problem of finding those $T_1$-spaces (Hausdorff spaces) which are densely embeddable in a pathwise connected $T_1$-space (Hausdorff space). In particular, we prove that a countable first countable Hausdorff space (with more than one point) is pathwise connectifiable (i.e., it can be densely embedded in a pathwise connected Hausdorff space) if and only if it has no isolated points. Moreover some examples are given to show that a Hausdorff space which can be densely embedded in a connected Hausdorff space may fail to be pathwise connectifiable. 

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## 0. Introduction

In the last years many papers have been devoted to the spaces which admit dense embeddings in connected spaces (see, e.g., [1,4,6,7]). This paper is the first contribution to the problem of densely embedding a $T_1$-space (Hausdorff space) in a pathwise connected $T_1$-space (Hausdorff space) (recall that a space $X$ is called pathwise connected if for every $x$ and $y$ in $X$ there is a continuous function $f: I \to X$ such that $f(0) = x$ and $f(1) = y$).

It is worth noting that every space can be embedded as a closed subspace of a pathwise connected space (see, e.g., [5,8]), but, as we shall see, the problem of finding dense embeddings is completely different.

After having characterized, in the first paragraph, those $T_1$-spaces which can be densely embedded in a pathwise connected $T_1$-space, we devote the main part of the paper to the investigation of the problem for $T_2$-spaces. We give some examples of nonpathwise

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connectifiable spaces and we show that every countable first countable Hausdorff space without isolated points is pathwise connectifiable.

We refer the reader to [3] for notations and terminology not explicitly given.

## 1. Dense embeddings in pathwise connected $T_1$-spaces

**Theorem 1.1.** *A $T_1$-space* (*with more than one point*) *can be densely embedded in a pathwise connected $T_1$-space if and only if it has no isolated points.*

**Proof.** If $X$ is a $T_1$-space (with more than one point) with an isolated point, then it cannot be even densely embedded in a connected $T_1$-space [7]. Now let $X$ be a $T_1$-space without isolated points. Let us show that $X$ can be densely embedded in a pathwise connected $T_1$-space $Z$.

Let $Y = X \times I$ and take a function $\lambda : X \to (0,1)$. We topologize $Y$ as follows. For every open subset $V$ of $X$ let $V(\lambda) = \{(x,\varepsilon)\colon\ x \in V,\ 0 \leqslant \varepsilon < \lambda(x)\}$. If $x \in X$ and $\kappa \in I$, then a basic neighbourhood of $(x,\kappa)$ in $Y$ has the following form:

$$\bigl(V(\lambda) \setminus \bigl(\{x\} \times [0,\lambda(x))\bigr)\bigr) \cup \{x\} \times I_\kappa,$$

where $V$ is an open neighbourhood of $x$ in $X$ and $I_\kappa$ is an open interval in $I$ containing $\kappa$. Now set $S = X \times \{1\}$ and let $Z$ be the quotient space obtained from $Y$ by identifying $S$ to a point. It is easy to see that $Z$ is a pathwise connected $T_1$-space in which $X \times \{0\}$ is a dense subspace homeomorphic to $X$. $\square$

**Remark 1.2.** A space $X$ is called hyperconnected if every pair of nonempty open subsets has nonempty intersection. It is clear that every hyperconnected $T_1$-space has no isolated points. Let us see that this kind of spaces can be densely embedded in a pathwise connected $T_1$-spaces in a different way. Let $(X,\tau)$ be a hyperconnected $T_1$-space and set $Z = X \cup Y$, where $Y$ is a set such that $|Y| = c$ and $X \cap Y = \emptyset$. Let $\sigma = \{\emptyset\} \cup \{(U \cup Y) \setminus F\colon\ U \in \tau \setminus \{0\},\ F \in [Y]^{<\omega}\}$. Since $X$ is hyperconnected, it follows that $\sigma$ is a topology on $Z$. Clearly $Z$ is $T_1$ and $X$ is a dense subspace of $Z$. Now observe that $Y \cup \{x\}$ is pathwise connected for every $x \in X$ (in fact $Y \cup \{x\}$ is a cofinite space of cardinality $c$). Therefore $Z$ is pathwise connected.

**Remark 1.3.** By the above theorem and Lemma 1.2 in [7] it follows that a $T_1$-space is densely embeddable in a pathwise connected $T_1$-space if and only if it is densely embeddable in a connected $T_1$-space. Observe that a $T_1$-space $X$ without isolated points can be densely embedded in a connected $T_1$-space $Z$ such that $|Z \setminus X| = 1$ (let $Z = X \cup \{\infty\}$, let the neighborhoods of $\infty$ be of the form $\{\infty\} \cup (X \setminus F)$ where $F \in [X]^{<\omega}$ and let $X$ be open in $Z$). Note that this cannot be done in our case, in fact every pathwise connected $T_1$-space (with more than one point) must have uncountable cardinality ($I$ cannot be written as the union of a countable family of pairwise disjoint nonempty closed sets).

## 2. Some spaces which are not pathwise connectifiable

A $T_2$-space which is densely embeddable in a pathwise connected $T_2$-space will be called pathwise connectifiable.

**Example 2.1.** Let $Y$ be a $T_2$-space which has a free open ultrafilter $p$ and let $X = Y \cup \{p\}$ be the space in which the neighborhoods of points of $Y$ are unchanged, and the neighborhoods of $p$ are the sets $G \cup \{p\}$ where $G \in p$. Then $X$ is not pathwise connectifiable.

Assume the contrary; then there is a pathwise connected $T_2$-space $Z$ such that $\overline{X} = Z$. Let $x_0 \in X$ and take a continuous injection $f : I \to Z$ such that $f(0) = x_0$ and $f(1) = p$. Let $\{t_n\colon n \in N\}$ be a sequence converging to 1 (with $t_n \neq 1$ for every $n$), clearly $f(t_n) = z_n \to p$. As $Z$ is Hausdorff we can construct, by induction, a subsequence $\{z_{n_\kappa}\}_\kappa$ of $\{z_n\}_n$ and two families $\{B_\kappa\colon \kappa \in N\}$ and $\{V_\kappa\colon \kappa \in N\}$ of open sets of $Z$ such that for every $\kappa$:

(i) $z_{n_\kappa} \in B_\kappa$ and $p \in V_\kappa$,

(ii) $B_\kappa \cap V_\kappa = \emptyset$,

(iii) $B_{\kappa+1} \subset V_\kappa$ and $V_{\kappa+1} \subset V_\kappa$.

Now set $H = \bigcup\{B_\kappa\colon \kappa = 2n,\ n \in N\} \cap Y$ and $K = \bigcup\{B_\kappa\colon \kappa = 2n + 1,\ n \in N\} \cap Y$. Since $H \cap K = \emptyset$, we will reach a contradiction if we show that $H, K \in p$. Since $p$ is an ultrafilter it is enough to show that $H$ and $K$ meet every member of $p$. Let $C \in p$. Since $C \cup \{p\}$ is open in $X$, there exists an open set $V$ of $Z$ such that $V \cap X = C \cup \{p\}$. Since $V$ is a neighbourhood of $p$ in $Z$, there is an even natural number $\kappa$ such that $z_{n_\kappa} \in V$ and therefore $B_\kappa \cap V \neq \emptyset$. Since $X$ is dense in $Z$, it follows that $(B_\kappa \cap V) \cap X \neq \emptyset$. Hence $\emptyset \neq (B_\kappa \cap X) \cap (V \cap X) = (B_\kappa \cap Y) \cap (C \cup \{p\}) \subset H \cap C$; therefore $H \in p$. In a similar way it can be shown that $K \in p$.

**Remark 2.2.** A space $Y$ is said to be extremally disconnected at the point $p$ if for every two disjoint open sets $U$ and $V$ in $Y$, $p \notin \overline{U} \cap \overline{V}$ (see, e.g., [2]). Observe that the proof of Example 2.1 can be used to show the following (more general) statement: a Hausdorff space with a point of extremal disconnectedness is not pathwise connectifiable.

**Remark 2.3.** Take $Y = Q$ and let $X = Q \cup \{p\}$ as in Example 2.1. Then $X$ can be densely embedded in a connected $T_2$-space (see Theorem 3.5 in [7]), but it is not pathwise connectifiable.

In the next example we show that a subspace of the Euclidean line can be densely embedded in a connected $T_2$-space without being pathwise connectifiable.

**Example 2.4.** For every $n \in N \cup \{0\}$ let $X_n = (1/(2n + 2), 1/(2n + 1)]$, and let us consider the following subspace of the euclidean line:

$$X = \{0\} \cup \bigcup \bigl\{X_n\colon n \in N \cup \{0\}\bigr\}.$$

Since $X$ is a metrizable space with no nonempty proper compact open subsets, it can be densely embedded in a connected $T_2$-space (see Corollary 2.10 of [6]). We claim that $X$ is not pathwise connectifiable.

Let us suppose that $Z$ is a pathwise connected $T_2$-space such that $X$ is dense in $Z$. For every $n \in N \cup \{0\}$ let $f_n : I \to Z$ be an embedding such that $f_n(0) = 1/(2n+1)$ and $f_n(1) = 0$. For every $n \in N \cup \{0\}$ let $t_n = \min\{t \in I\colon\ f_n(t) \notin X_n\}$ (it is easy to see that $t_n$ exists) and set $z_n = f_n(t_n)$.

Since $f_n$ is an embedding, there is some $c \geqslant 1/(2n+2)$ such that $f_n([0,t_n)) = (c, 1/(2n+1)]$. We claim that $c = 1/(2n+2)$ (and hence $f_n([0,t_n)) = X_n$). Suppose that $c > 1/(2n+2)$, then $z_n = c \in X_n$ (by the continuity of $f_n$), and we reach a contradiction. Therefore $\mathrm{cl}_Z(X_n) = X_n \cup \{z_n\}$ for every $n \in N \cup \{0\}$.

First let us show that $Z = X \cup \{z_n\colon\ n \in N \cup \{0\}\}$. Suppose there is some $z \in Z \setminus (X \cup \{z_n\colon\ n \in N \cup \{0\}\})$. Since $z \neq 0$, there are disjoint open subsets $U$ and $V$ of $Z$ such that $z \in U$ and $0 \in V$. Take $\kappa \in N$ such that $\bigcup\{X_n\colon\ n \geqslant \kappa\} \subset V$, obviously $\mathrm{cl}_Z(\bigcup\{\mathrm{cl}_Z(X_n)\colon\ n \geqslant \kappa\}) \subset \mathrm{cl}_Z(V)$ and $\mathrm{cl}_Z(V) \cap U = \emptyset$. Since $z \notin \bigcup\{\mathrm{cl}_Z(X_n)\colon\ n \in \{0,\dots,\kappa-1\}\}$, there is an open neighbourhood $W$ of $z$ in $Z$ such that

$$W \cap \bigcup\bigl\{\mathrm{cl}_Z(X_n)\colon\ n \in \{0,\dots,\kappa-1\}\bigr\} = \emptyset.$$

Therefore $G = U \cap W$ is a nonempty open subset of $Z$ such that $G \cap X = \emptyset$, a contradiction.

Now let $A = \{i \in N \cup \{0\}\colon\ z_i = z_0\}$ and set $K = \bigcup\{\mathrm{cl}_Z(X_i)\colon\ i \in A\}$. Obviously, as $0 \notin K$, $K$ is a proper nonempty subset of $Z$. We reach a contradiction if we show that $K$ is clopen in $Z$. Since $z_0 \neq 0$, there are disjoint open subsets $U$ and $V$ of $Z$ such that $z_0 \in U$ and $0 \in V$. Take $\kappa(V) \in N$ such that $\bigcup\{X_i\colon\ i \geqslant \kappa(V)\} \subset V$, and hence $\mathrm{cl}_Z(\bigcup\{\mathrm{cl}_Z(X_i)\colon\ i \geqslant \kappa(V)\}) \subset \mathrm{cl}_Z(V)$. Since $\mathrm{cl}_Z(V) \cap U = \emptyset$, it follows that $z_i \neq z_0$ for every $i \geqslant \kappa(V)$. Therefore $A$ is finite and $K$ is closed in $Z$. Before showing that $K$ is open in $Z$ let us see that every $X_n$ is open in $Z$. Let $Y = X \setminus X_n$. Since $X_n$ is open in $X$, there is an open subset $W$ of $Z$ such that $W \cap X = X_n$. Observe that $W \cap \mathrm{cl}_Z(Y) = \emptyset$. Now $Z = \mathrm{cl}_Z(X_n \cup Y) = X_n \cup \{z_n\} \cup \mathrm{cl}_Z(Y)$, hence $W \subset X_n \cup \{z_n\}$. So $X_n \subset W \subset X_n \cup \{z_n\}$, therefore $X_n = W$ or $X_n = W \setminus \{z_n\}$. Hence $X_n$ is open in $Z$.

Now we can show that $K$ is open in $Z$. Let $p \in K$; since every $X_n$ is open in $Z$, it is enough to consider the case in which $p = z_0$. For every $j < \kappa(V)$ such that $z_j \neq z_0$ take an open neighbourhood $U_j$ of $z_0$ in $Z$ such that $U_j \cap \mathrm{cl}_Z(X_j) = \emptyset$. Since $U$ is an open set of $Z$ containing $z_0$ such that $U \cap \mathrm{cl}_Z(X_j) = \emptyset$ for every $j \geqslant k(V)$, it follows that $U \cap \bigcap\{U_j\colon\ j < \kappa(V),\ z_j \neq z_0\}$ is an open neighbourhood of $z_0$ in $Z$ which is contained in $K$.

Example 2.4, and every continuum in the euclidean plane which is not pathwise connected, show that a metrizable second countable space with no nonempty proper compact open subsets may fail to be pathwise connectifiable. So it is natural to ask the following:

**Problem 2.5.** Under which conditions a metrizable second countable space is pathwise connectifiable?

## 3. Pathwise connectifiable spaces

Remark 2.3 shows that a countable Hausdorff space without isolated points may fail to be pathwise connectifiable.

**Theorem 3.1.** *A countable first countable Hausdorff space $X$ (with more than one point) is pathwise connectifiable if and only if it has no isolated points.*

**Proof.** The necessity is clear. So let us prove the sufficiency. Let $\tau$ be the topology on $X$ and assume that the underlying set of $X$ is $N$. For every $n \in N$ let $\mathcal{B}_n = \{B(n,i)\colon i \in N\}$ be a decreasing base for $X$ at the point $n$ such that $C(n,i) = B(n,i) \setminus \overline{B(n,i+1)} \neq \emptyset$ for every $i \in N$ (this can be done because $X$ is a Hausdorff space without isolated points). Observe that for every $n \in N$ it follows that $C(n,i) \cap C(n,j) = \emptyset$ whenever $i \neq j$. If $A$ is a nonempty open set of $X$, $\mathcal{G}(A)$ will denote the family of free open filters $\mathcal{F}$ on $X$ with a countable base such that $A \in \mathcal{F}$. $\square$

Now the proof of the theorem will proceed in several steps.

**Claim 1.** *$\mathcal{G}(A) \neq \emptyset$ for every nonempty open subset $A$ of $X$.*

**Proof.** Let $F_1 = A$ and suppose to have constructed $F_1, \ldots, F_n$ in such a way that:
(i) $F_i$ is a nonempty open set of $X$ for every $i \leqslant n$,
(ii) $F_i \subset F_j$ if $j < i \leqslant n$,
(iii) for every $i \leqslant n$ there is a $B_i \in \mathcal{B}_i$ such that $B_i \cap F_i = \emptyset$.

Pick a point $x \in F_n$ such that $x \neq n+1$, and take a $B \in \mathcal{B}_{n+1}$ and a $H \in \mathcal{B}_x$ such that $H \subset F_n$ and $B \cap H = \emptyset$. Set $F_{n+1} = H$ and $B_{n+1} = B$. This completes the inductive construction. Clearly the open filter $\mathcal{F}$ generated by $\{F_n\colon n \in N\}$ is an element of $\mathcal{G}(A)$.

A collection $\{\mathcal{C}_\lambda\colon \lambda \in \Lambda\}$ of families of subsets of $X$ is called totally disjoint if for every $\lambda \in \Lambda$ there is a $C_\lambda \in \mathcal{C}_\lambda$ such that $\{C_\lambda\colon \lambda \in \Lambda\}$ is a family of pairwise disjoint sets. $\square$

**Claim 2.** *Let $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{G}(N)$ and let $H$ be a nonempty open subset of $X$. Then there is some $\mathcal{H} \in \mathcal{G}(H)$ such that $\{\mathcal{H}, \mathcal{H}_i\}$ is totally disjoint for every $i \leqslant n$.*

**Proof.** Let $x \in H$; then there exist $B \in \mathcal{B}_x$ and $H_i \in \mathcal{H}_i$ such that $B \cap H_i = \emptyset$ for every $i \leqslant n$. Pick $\mathcal{H} \in \mathcal{G}(H \cap B)$; then $\mathcal{H}$ has the required properties.

Applying Claim 2 it is possible to choose, for every $n, i \in N$, a filter $\mathcal{F}(n,i) \in \mathcal{G}(C(n,i))$ in such a way that $\{\mathcal{F}(n,i), \mathcal{F}(m,j)\}$ is totally disjoint whenever $(n,i) \neq (m,j)$. $\square$

**Claim 3.** *Let $\mathcal{H} \in \mathcal{G}(N)$. Then for every $n \in N$ there is an open filter $\mathcal{H}_n$ with a countable base such that $\mathcal{H} \subset \mathcal{H}_n$ and $\{\mathcal{H}_n\colon n \in N\}$ is totally disjoint.*

**Proof.** Let $\{H_1 \supset H_2 \supset \cdots\}$ be a filter base of $\mathcal{H}$ with $C_i = H_i \setminus \overline{H}_{i+1} \neq \emptyset$. Take a partition $\{K_n\colon n \in N\}$ of $N$ such that $|K_n| = \omega$ for every $n$.

Now for every $n, i \in N$ set $H(n,i) = \bigcup\{C_j\colon\ j > i,\ \ j \in K_n\}$ and let $\mathcal{H}_n$ be the filter generated by $\{H(n,i)\colon\ i \in N\}$. Then the family $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$ has the required properties. $\square$

Now we are able to start the construction of a Hausdorff pathwise connected space $Z$ in which $X$ can be densely embedded.

Apply Claim 3 in order to obtain, for every $n, i \in N$, a sequence $\{\mathcal{F}(n,i,\kappa)\colon\ \kappa \in N\}$ of open filters with a countable base so that $\mathcal{F}(n,i) \subset \mathcal{F}(n,i,\kappa)$ for every $\kappa$ and $\{\mathcal{F}(n,i,\kappa)\colon\ \kappa \in N\}$ is totally disjoint.

Moreover from Claim 3 and the choice of the $F(n,i,k)$ we can deduce the following

**Claim 4.** *For every $m$ the family $\{\mathcal{F}(n,i,\kappa)\colon\ n, i \leqslant m,\ \kappa \in N\}$ is totally disjoint.*

Let $Q_+ = Q \cap [0,\infty)$ and $R_+ = [0,\infty)$ For every $n, i \in N$ take a bijection

$$\phi_{n,i} : \left\{x \in Q_+\colon\ \frac{1}{i+1} \leqslant |x-n| < \frac{1}{i}\right\} \to N$$

and let $\sigma$ be the topology on $R_+$ given by the following neighbourhood system:

(i) $x = q \in Q_+ \setminus \{n/2\colon\ n \in N\}$. Let $n_q = [q + \frac{1}{2}]$ (the greatest integer function) and let $i_q \in N$ such that $1/(i_q+1) \leqslant |q - n_q| < 1/i_q$, set $\kappa_q = \phi_{n_q,i_q}(q)$. A neighbourhood of $q$ is of the form $F \cup \{q\}$, where $F \in \mathcal{F}(n_q, i_q, \kappa_q)$.

(ii) $x = n \in N$. A neighbourhood of $n$ is any $B$ with $B \in \mathcal{B}_n$.

(iii) Every other point of $R_+$ is isolated.

Now let $\tau_E$ be the euclidean topology on $R_+$, set $\lambda = \sigma \cap \tau_E$ and $Z = (R_+, \lambda)$.

We claim that

(1) $Z$ is pathwise connected;

(2) $X$ is a dense subspace of $Z$;

(3) $Z$ is a Hausdorff space.

Since $\lambda$ is coarser than the euclidean topology on $R_+$, it follows that $Z$ is pathwise connected.

Now let $\lambda|N = \{G \cap N\colon\ G \in \lambda\}$ and let us show that $\tau = \lambda|N$ (i.e., $X$ is a subspace of $Z$). Observe that $\lambda|N \subset \sigma|N = \tau$, so it remains to show that $A \in \lambda|N$ for every $A \in \tau$. Let $A \in \tau$ and set $A^* = \bigcup\{(n - 1/i_n, n + 1/i_n)\colon\ n \in A\}$ where $i_n = \min\{i \geqslant 2\colon\ B(n,i) \subset A\}$, obviously $A^* \in \tau_E$ and $A \subset A^*$. We claim that $A^* \in \sigma$ (and therefore $A^* \in \lambda$). Take $q \in A^*$ (it is enough to consider the case in which $q \in (Q_+ \setminus \{n/2\colon\ n \in N\})$): let $n$ be such that $q \in (n - 1/i_n, n + 1/i_n)$, set $\kappa = \phi_{n,i}(q)$, where $i = i_q$ and take $B(n,i) \in \mathcal{F}(n,i,\kappa)$. Observe that $B(n,i) \subset B(n,i_n)$ $(i \geqslant i_n)$ and $B(n,i_n) \subset A$, therefore $B(n,i) \cup \{q\}$ is a neighbourhood of $q$ in $(R_+, \sigma)$ which is contained in $A^*$. Therefore $A^* \in \sigma \subset \lambda$. Since $A = A^* \cap N$, it follows that $A \in \lambda|N$.

Now let us show that $X$ is dense in $Z$. Let $G$ be a nonempty open subset of $Z$ and take a $q \in G \cap Q$. Since $G \in \sigma$, there is some $F \in \mathcal{F}(n_q, i_q, \kappa_q)$ such that $F \cup \{q\} \subset G$. Therefore $\emptyset \neq F \cap N \subset G \cap N$.

It remains to show that $Z$ is a Hausdorff space. Let $x$ and $y$ be two distinct points of $Z$.

(i) $x, y \in N$ (let us say $x = m$ and $y = n$). Since $X$ is $T_2$, there are $B_x \in \mathcal{B}_m$ and $B_y \in \mathcal{B}_n$ such that $B_x \cap B_y = \emptyset$. It is enough to observe that $B_x^*$ and $B_y^*$ are disjoint open subsets of $Z$ with $x \in B_x^*$ and $y \in B_y^*$.

(ii) $x, y \notin N$. Take two positive numbers $\varepsilon_x$ and $\varepsilon_y$ so that $I_x = (x - \varepsilon_x, x + \varepsilon_x)$ and $I_y = (y - \varepsilon_y, y + \varepsilon_y)$ are disjoint and $I_x \cap N = I_y \cap N = \emptyset$. Let $S = (Q_+ \setminus \{n/2\colon n \in N\}) \cap (I_x \cup I_y)$ and observe that $\{\mathcal{F}(n_q, i_q, \kappa_q)\colon q \in S\}$ is totally disjoint (by Claim 4). So, for every $q \in S$, there is a $B_q \in \mathcal{F}(n_q, i_q, \kappa_q)$ such that the family $\{B_q\colon q \in S\}$ is cellular. Set $B_q = \emptyset$ for every $q \in (I_x \cup I_y) \setminus S$, $A_x = \bigcup\{B_q\colon q \in I_x\}$ and $A_y = \bigcup\{B_q\colon q \in I_y\}$. Now let $G_x = I_x \cup A_x$ and $H_y = I_y \cup A_y$. For every $n \in A_x$ take $\varepsilon_n > 0$ such that $(n - \varepsilon_n, n + \varepsilon_n) \cap I_y = \emptyset$ and $\varepsilon_n < 1/i_n$ (see the definition of $A^*$), moreover for every $m \in A_y$ take $\delta_m > 0$ such that $(m - \delta_m, m + \delta_m) \cap I_x = \emptyset$ and $\delta_m < 1/i_m$. Let $G = I_x \cup \bigcup\{(n - \varepsilon_n, n + \varepsilon_n)\colon n \in A_x\}$ and $H = I_y \cup \bigcup\{(m - \delta_m, m + \delta_m)\colon m \in A_y\}$. Obviously $x \in G$ and $y \in H$, moreover it is straightforward to check that $G$ and $H$ are disjoint open subsets of $Z$.

(iii) $x = m \in N$ and $y \notin N$. Take a positive number $\varepsilon$ such that $I_y = (y - \varepsilon, y + \varepsilon)$ does not meet $N$. Let $S = (Q_+ \setminus \{n/2\colon n \in N\}) \cap I_y$, since $\{\mathcal{F}(n_q, i_q)\colon q \in S\}$ is finite, it follows that the family $\{\mathcal{F}(n_q, i_q)\colon q \in S\} \cup \{\mathcal{B}_m\}$ is totally disjoint. So from Claim 4, $\{\mathcal{F}(n_q, i_q, \kappa_q)\colon q \in S\} \cup \{\mathcal{B}_m\}$ is totally disjoint too.

Now take $B_q \in \mathcal{F}(n_q, i_q, \kappa_q)$ and $B \in \mathcal{B}_m$ so that $\{B_q\colon q \in S\} \cup \{B\}$ is cellular. Let $B_q = \emptyset$ for every $q \in I_y \setminus S$ and set $A_x = B$ and $A_y = \bigcup\{B_q\colon q \in I_y\}$. Arguing as in the previous case (with $I_x = \emptyset$) we can obtain two disjoint open subsets $G$ and $H$ of $Z$ such that $x \in G$ and $y \in H$. Therefore $Z$ is a Hausdorff space.

We conclude this paper with the following general

**Problem 3.2.** Characterize those Hausdorff spaces which are pathwise connectifiable.

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